SECTION – II

MEASURABLE FUNCTIONS

Measurable Function: An extended real valued function **f** defined on a measurable set E is said to be measurable function if {x| f(x) $>\ \alpha\}$ is measurable for each real number α.

2.1 Theorem. A constant function with a measurable domain is measurable.

Proof: Let **f** be a constant function with a measurable domain E and Let **f** : E→ R be a constant function i.e., **f**(x) = k ∀ $x\ \in E$ $and\ k\ is\ constant$.

To show that {x| f(x) $>\ \alpha\}$ is measurable for each real number α.

$$\{x|\ \mathbf{f}(x) > \alpha\} = \begin{cases} E, & k > \alpha \\ \varphi, & k = \alpha \\ \varphi, & k < \alpha \end{cases}$$

Since both φ and E are measurable, it follows that the set {x| **f**(x) $>\ \alpha\}$ and hence **f** is measurable.

2.2 Theorem. Let f be an extended real valued function defined on a measurable set E, Then f is said to be measurable (Lebesgue function) if for any real α any one of the following four conditions is satisfied.

(a) {x| f(x) $>\ \alpha\}$ is measurable

(b) {x| f(x) $\geq\ \alpha\}$ is measurable

(c) {x| f(x) $<\ \alpha\}$ is measurable

(d) {x| f(x) $\leq \alpha\}$ is measurable.

Proof: We show that these four conditions are equivalent. First of all we show that (a) and (b) are equivalent. Since

$$\{x|\ f(x) > \alpha\} = \{x|\ f(x) \leq \alpha\}^c$$

And also we know that complement of a measurable set is measurable, therefore (a) ⟹(d) and conversely.

Similarly since (b) and (c) are complement of each other, (c) is measurable if (b) is measurable and conversely.

Therefore, it is sufficient to prove that (a) ⟹(b) and conversely.

Firstly we show that (b) ⟹ (a).

The set {x| f(x) $\geq\ \alpha\}$ is given to be measurable.

Now

$$\{x \mid f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \mid f(x) \geq \alpha + \frac{1}{n}\}$$

But by (b), $\{x \mid f(x) \geq \alpha + \frac{1}{n}\}$ is measurable. Also we know that countable union of measurable sets is measurable. Hence $\{x \mid f(x) > \alpha\}$ is measurable which implies that (b) ⟹(a).

Conversely, let (a) holds. We have

$$\{x \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) \geq \alpha - \frac{1}{n}\}$$

The set $\{x \mid f(x) > \alpha - \frac{1}{n}\}$ is measurable by (a). Moreover, intersection of measurable sets is also measurable . Hence $\{x \mid f(x) \geq \alpha\}$ is also measurable . Thus (a) ⟹ (b).

Hence the four conditions are equivalent.

2.3 Remark: We can say that f is measurable function if for any real number α, any of the four conditions in the above theorem holds.

2.4 Lemma. If α is an extended real number then these four conditions imply that $\{x \mid f(x) = \alpha\}$ is also measurable.

Proof. Let α be a real number, then

$$\{x \mid f(x) = \alpha\} = \{x \mid f(x) \geq \alpha\} \cap \{x \mid f(x) \leq \alpha\}.$$

Since $\{x \mid f(x) \geq \alpha\}$ and $\{x \mid f(x) \leq \alpha\}$ are measurable by conditions (b) and (d), the set $\{x \mid f(x) = \alpha\}$ is measurable being the intersection of measurable sets.

Suppose $\alpha = \infty$. Then $\{x \mid f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > n\}$

Which is measurable by the condition (a) and the fact intersection of measurable sets is measurable.

Similarly when $= -\infty$, then

$\{x \mid f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) < -n\}$, which is again measurable by conditions (c).Hence the results follows.

2.5 Theorem: If f is measurable function on each of the sets in a countable collection $\{E_i\}$ of disjoint measurable sets, then f is measurable on E = $\bigcup_i E_i$.

Proof: Let E = $\bigcup_i E_i$. Then E is measurable being countable union of measurable sets is measurable.

Let α be any real number.

Consider the set $\{x \in E \mid f(x) > \alpha\} = \bigcup_i \{x \in E_i : f(x) > \alpha\}$ is measurable.

Because f is measurable on each E_i.

⇨ $\bigcup_i \{x \in E_i : f(x) > \alpha\}$ is measurable.

⇨ {x∈ E| f(x) $>$ α} is measurable.
Hence f is measurable on E.

2.6 Theorem: If f is measurable function on E and $E_1 \subseteq E$ is measurable set then f is a measurable function on E_1.

Proof: Let α be any real number.

Consider the set {x∈ E_1| f(x) $>$ α} = {x∈ E| f(x) $>$ α}∩ E_1 is measurable.

2.7 Theorem. If f and g are measurable functions on a common domain E, then the set A= {x∈ E: f(x)< g(x)} is measurable.

Proof. For each rational number r, define

$$A_r = \{ x\in E: f(x) < r < g(x)\}$$

Or we can write

$$A_r = \{ x\in E: f(x)< r\}\cap \{x\in E: g(x)> r\}$$

Since f and g are measurable on E, so the two sets on R.H.S. are measurable sets is measurable.

Now, we observe that

$$\{x\in E: f(x) <g(x)\} = \cup_{r\in Q} A_r$$

Since the rationals are countable, so A is countable union of measurable sets and so is measurable.

This proves the theorem.

2.8 Theorem. A continuous function defined on a measurable set is measurable.

Proof. Let f be a continuous function defined on measurable set E. Let α be any real number. We now claim that {x ∈ E : f(x) ≥ α } is closed.

Let A= { x ∈ E: f(x) ≥ α } (1)

To prove that A is closed, it is sufficient to show that $A' \subseteq A$. (2)

A' being derived set of A.

Let $x_0 \in A'$ be arbitrary element. Then $x_0 \in A'$ implies x_0 is limit point of A.

It implies that there exist a sequence $\{x_n\}$ whose elements $x_n \in A$ such that

$$\lim_{n\to\infty} x_n = x_0$$

Moreover, f is continuous at x_0; it follows that by definition of continuity $x_n \to x_0$ implies $f(x_n)\to f(x_0)$ (3)

By (2), we see that x_n ∈ A for all n ∈ N.

⇒ f(x_n) ≥ α for all n∈ N.

⇒ $\lim_{n\to\infty} f x_n \geq \alpha$

$\Rightarrow f(x_0) \geq \alpha$ by virtue of (3)

$\Rightarrow x_0 \in A$ by (1)

Further any $x_0 \in A'$ implies $x_0 \in A$

$\Rightarrow A' \subseteq A$

⇒ A is closed

⇒A is measurable

⇒{x ∈ E: f(x) ≥ α } is measurable.

Hence a continuous function f is measurable on E.

Converse of above theorem is not true, that is, A measurable function need not be continuous.

2.9 Example. Consider a function f: R → [0,1]

defined by f(x) = $\begin{cases} \mathbf{1} & \boldsymbol{if\ 0 \leq x \leq 1} \\ \mathbf{0} & \boldsymbol{if\ otherwise} \end{cases}$ **.**

Clearly function is not continuous since 0 is the point of discontinuity.

For any real number α,

$$\{x \in N: f(x) > \alpha \} = \begin{cases} \varphi, & \alpha \geq 1 \\ R, & \alpha < 0 \\ [0,1), & \alpha \leq 0 < 1 \end{cases}$$

Since R, φ, [0,1) are measurable implies f is measurable function on E.

2.10 Theorem. Let f be a function defined on a measurable set E then f is measurable iff for any open set G in R the inverse image f^{-1} (G) is measurable set.

Proof. Let f be a measurable function and let G be any set in R. Since every open sets can be written as countable union of disjoint open intervals.

Suppose, G = $\cup_n I_n = \cup_n (a_n, b_n)$

Then $f^{-1}(G) = f^{-1}(\cup_n I_n) = \cup_n \{ f^{-1}(I_n)\}$

$$= \cup_n\{ x: f(x) \in I_n\}$$

$$= \cup_n \{x : f(x) \in (a_n, b_n)\}$$

but $\{x : f(x) \in (a_n, b_n)\} = \{x : a_n < f(x) < b_n\}$

$$= \{x : f(x) > a_n\} \cap \{x : f(x) < b_n\}$$

Since f is measurable function. So both sets on R.H.S. are measurable and hence

$\{x : f(x) \in (a_n, b_n)\}$ is measurable.

Again f^{-1} (G) is measurable. [since countable union of measurable sets is measurable]

Conversely:- Let f^{-1} (G) be measurable for every open set G in R. We have to prove that f is measurable function.

Take G = (α, ∞) where α is any real no.

Then $f^{-1}(\alpha, \infty)$ is measurable

that is, $\{x : f(x) \in (\alpha, \infty)\}$ is measurable

that is , $\{x : f(x) > \alpha\}$ is measurable.

Thus f is measurable function.

2.11 Theorem. Let f be continuous and g be measurable function then fog is measurable.

Proof. Let α be any real number then

$$\{x : fog(x) > \alpha\} = \{x : f(g(x)) > \alpha\}$$
$$= \{x : f(g(x)) \in (\alpha, \infty)\}$$
$$= \{x : g(x) \in f^{-1}(\alpha, \infty)\}.$$

Now, (α, ∞) is open subset of R and f is continuous implies $f^{-1}(\alpha, \infty)$ is open set.

Hence, it can be written as countable union of disjoint open intervals say

$$f^{-1}(\alpha, \infty) = \cup_n I_n = \cup_n (a_n, b_n).$$

Therefore,

$$\{x : fog(x) > \alpha\} = \{x : g(x) \in \cup_n I_n\} = g^{-1}(\cup_n I_n)$$
$$= \cup_n g^{-1}(I_n)$$
$$= \cup_n \{x : g(x) \in I_n\}$$
$$= \cup_n \{x : g(x) \in (a_n, b_n)\}$$
$$= \cup_n \{x : a_n < g(x) < b_n)\}$$
$$= \cup_n \{x : g(x) > a_n)\} \cap \cup_n \{x : g(x) < b_n)\}.$$

Since g is measurable function. Both sets on R.H.S. are measurable and their intersection is measurable. Also countable union of measurable sets is measurable. Hence the result.

2.12 Definition. A function f is said to be a step function iff

f(x) = Ci , $\xi_{i-1} < x < \xi_i$ for some subdivision of [a, b] and some constants Ci .

Example: A function f : [0, 1] → R defined as f(x) = $\begin{cases} \alpha, & a \le x \le c \\ \beta, & c \le x \le b \end{cases}$ where α, β are constant, f is a step function.

Remark: Every step function is a measurable function.

2.13 Theorem. For any real no c and two measurable real- valued functions, f and g, the functions f +c, cf, f+g, f-g, fg and f/g (g$\neq$ 0), $|f|$ are all measurable.

Proof. We are given that f is measurable function and c ais any real number. Then for any real number α

$$\{x|\ f(x) + c > \alpha\} = \{x|\ f(x) > \alpha - c\}$$

But $\{x|\ f(x) > \alpha - c\}$ is measurable by the condition (a) of the definition. Hence

$\{x|\ f(x) + c > \alpha\}$ and so | f(x) $+c$ is measurable.we next consider the function cf . in case c= 0, cf is the constant function 0 and hence is measurable since every constant function is continuous and so measurable. In case c $>$ 0 we have

$$\{x|\ cf(x) > \alpha\} = \{x|\ f(x) > \frac{\alpha}{c}\}, \text{ and so measurable.}$$

In case c $<$ 0, we have $\{x|\ cf(x) > r\} = \{x|\ f(x) < \frac{r}{c}\}$ and so measurable.

Now if f and g are two measurable real valued functions defined on the same domain, we shall show that f+g is measurable. To show that it is sufficient to show that the set

$\{x|\ f(x)+g(x) > \alpha\}$ is measurable.

if f(x) +g(x)$>$ α, then | f(x) $> \alpha - g(x)$ and by he cor. of the axiom of Archimedes there is a rational number r such that $\alpha - g(x) < r < f(x)$

since the functions f and g are measurable , the sets $\{x|\ f(x) > r\}$ and $\{x|\ f(x) > \alpha - r\}$ are measurable. Therefore, there intersection $S_r = \{x|\ f(x) > \alpha - c\} \cap \{x|\ f(x) > \alpha - r\}$ also measurable.

It can be shown that $\{x|f(x)+g(x) > \alpha\} = \cup \{S_r \mid r \text{ is rational}\}$

Since the set of rational is countable and countable union of measurable sets is measurable , the set $\cup \{S_r \mid r \text{ is rational}\}$ and hence $\{x|f(x)+g(x) > \alpha\}$is measurable which proves that

f(x) +g(x) is measurable. From this part it follows that f- g = f (-g) is also measurable, since when g is measurable (-g) is also measurable. Next we consider fg.

The measurability of fg follows that from the identity fg = $\frac{1}{2}[(f+g)^2 - f^2 - g^2]$, if we prove that f^2 is measurable when f is measurable. For this it is sufficient to prove that

$\{x|\in E|f^2(x) > \alpha\}$, α is real number, is measurable.

Let α be a negative real number. Then it is clear that the set $\{x|f^2(x) > \alpha\}$ = E (domain of the measurable function f). But E is measurable by the definition of f. Hence $\{x|f^2(x) > \alpha\}$ is measurable when $\alpha < 0$.

Now let $\alpha \geq 0$, then $\{x|f^2(x) > \alpha\} = \{x|\ f(x) > \sqrt{\alpha}\} \cup \{x|\ f(x) < -\sqrt{\alpha}\}$.

Since f is measurable, it follows from this equality that $\{x|f^2(x) > \alpha\}$ is measurable for $\alpha \geq 0$.

Hence f^2 is also measurable when f is also measurable. Therefore, the theorem follows from the above identity, since measurability of f and g imply the measurability of f+g.

Consider $\frac{f}{g}(g\neq 0) = f.\frac{1}{g}$

First we have to prove that $\frac{1}{g}$ is measurable.

Consider the set $\left\{x : \left(\frac{1}{g}\right)(x) > \alpha\right\} = \left\{x : \frac{1}{g(x)} > \alpha\right\}$

$$= \begin{cases} \{x: g(x) > 0\} if\ \alpha = 0 \\ \{x: g(x) > 0\} \cap \left\{x: g(x) < \frac{1}{\alpha}\right\} if\ \alpha > 0 \\ \{x: g(x) > 0\} \cup \left\{\{x: g(x) > 0\} \cap \left\{x: g(x) < \frac{1}{\alpha}\right\}\right\} if\ \alpha < 0 \end{cases}$$

Since g is measurable in each case ,i.e., $\left\{x : \left(\frac{1}{g}\right)(x) > \alpha\right\}$is measurable.

⇨ $\frac{1}{g}$ is measurable.

Since f and $\frac{1}{g}$ are measurable.

⇨ $\frac{f}{g}$ is measurable.

Now If f is measurable then |f| is also measurable.

It suffices to prove that measurability of the set {x| f(x) $>\ \alpha\}$= E (domain of f)

But E is assumed to be measurable. Hence {x| f(x) $>\ \alpha\}$={x| f(x) $>\ \alpha\}\cup${x| f(x)$<\ -\alpha$ }

The right hand side of the equality is measurable since f is measurable. Hence {x| f(x) $>\ \alpha\}$is measurable. Hence |f| is measurable.

2.14 Remark: Converse of (vii) is not true.

Example: Let P be a non-measurable subset of [0, 1) = E

Define a function f : E→ R as

f(x) = $\begin{cases} \boldsymbol{1\ \ if\ x \in P} \\ \boldsymbol{-1\ \ if\ x\ not\ belongs\ to\ P} \end{cases}$

⇨ f is not measurable because $\{x: f(x) > 0\} = \{x: f(x) = 1\} = P\ which\ is\ non-measurable.$

Also, for any real $\alpha, \{x: |f|(x) > \alpha\} = \{x: |f(x)| > \alpha\} = \begin{cases} \varphi, if\ \alpha \geq 1 \\ E, if\ \alpha < 1 \end{cases}$

Since E and φ are measurable.

⇨ $\{x: |f|(x) > \alpha\}\ is\ measurable.$

2.15 Theorem. Let $\{f_n\}_{n=1}^{\infty}$ **be a sequence of measurable functions. Then sup{f₁,f₂,…,fₙ}, inf {f₁,f₂,…,fₙ},** $\sup_n, \inf_n, \lim_n f_n and \lim f_n$ **are measurable.**

Proof. Define a function M(x) = $sup\{f_1(x), f_2(x), ..., f_n(x)\}$ we shall show that

{x| M(x)> $\alpha\}$is measurable. In fact {x| M(x)> $\alpha\}= \cup_i^n\{x: f_i(x) > \alpha\}$

Since each f_i is measurable, each of the set $\{x|\ f_i(x) > \alpha\}$ is measurable and therefore their union is also measurable. Hence $\{x|\ M(x) > \alpha\}$ and so M(x) is measurable. Similarly we define the function m(x) = inf $\{f_1, f_2,,,, f_n\}$, since $\{x|\ m(x) < \alpha\} = \cup_i^n \{x: f_i(x) < \alpha\}$ and

since $\{x|\ f_i(x) < \alpha\}$ is measurable on account of the the measurability of f_i, it follows that $\{x|\ m(x) < \alpha\}$ and so m(x) is measurable . Define a function M'(x) $= \sup_n f_n(x) = \sup\{f_1, f_2,,,, f_n\}$

We shall show that the set $\{x|\ M'(x) > \alpha\}$ is measurable for any real α.

Now $\{x|\ M'(x) > \alpha\} = \cup_{n=1}^{\infty} \{x: f_n(x) > \alpha\}$ is measurable, since each f_n is measurable.

Similarly if we define m'(x) $= \inf_n f_n(x)$, then $\{x|\ m'(x) < \alpha\} = \cup_{n=1}^{\infty} \{x: f_n(x) < \alpha\}$ and therefore measurability of f_n implies that of m'(x). Now since $\overline{lim} f_n = \inf_n \sup_{k \geq n} f_k$ and

$\underline{lim} f_n = \sup_n \inf_{k \geq n} f_k$, the upper and lower limit are measurable.

2.16 Corollary: If $\{f_n\}$ is a sequence of measurable functions converging to f. Then f is also measurable.

Proof: Since $\{f_n\}$ converges to f ,i.e., $\lim_{n\to\infty} f_n = f$

Then $\overline{lim} f_n = \underline{lim} f_n = \lim_{n\to\infty} f_n$

i.e., f = $\overline{lim} f_n = \underline{lim} f_n$

Hence f is measurable because $\overline{lim} f_n$ *and* $\underline{lim} f_n$ *are measurable.*

2.17 Corollary: The set of points on which a sequence $\{f_n\}$ of measurable functions converges is measurable.

Proof: By above theorem $\overline{lim} f_n$ *and* $\underline{lim} f_n$ *are measurable.*

⇨ $\overline{lim} f_n - \underline{lim} f_n$ *is measurable.*

Therefore, $\{x: [\overline{lim} f_n - \underline{lim} f_n](x) = \alpha\}$ *is measurable* $\forall \alpha$.

In Particular, for $\alpha = c$

$$\{x: [\overline{lim} f_n - \underline{lim} f_n](x) = 0\} \text{ is measurable.}$$

i.e.,

$$\{x: [\overline{lim} f_n(x) = \underline{lim} f_n(x)]\} \text{ is measurable.}$$

i.e., set of these points for which $\{f_n\}$ converges is measurable.

2.18 Definition. Let f and g be measurable functions. Then we define

f^+ = Max (f, 0)

f^- = Max (-f, 0)

f $\vee$ g = $\frac{f+g+|f-g|}{2}$ i.e. Max (f, g) and

f $\wedge$ g = $\frac{f+g-|f-g|}{2}$ i.e. min (f, g)

2.19 Theorem. Let f be a measurable function. Then $\overset{+}{f}$ and f^- are both measurable.

Proof. Let us suppose that f > 0 . Then we have

$$f = \overset{+}{f} - f^- \qquad \text{(i)}$$

Now let us take f to be negative.

Then $\overset{+}{f} =$ Max (f, 0) = 0, (ii)

$\overset{+}{f}$ = Max (-f, 0) = -f

Therefore on subtraction $f = \overset{+}{f} - f^-$

In casef = 0 , then $\overset{+}{f} = 0$, $f^- = 0$. (iii)

Therefore $f = \overset{+}{f} - f^-$

Thus for all f we have, $f = \overset{+}{f} - f^-$

Also adding the components of (i) we have

$$f = |f| = \overset{+}{f} + f^- \qquad \text{(v)}$$

since f is positive. And from (ii) when f is negative we have

$$\overset{+}{f} + f^- = 0 - f^- = f^- = |f| \qquad \text{(vi)}$$

In case f is zero, then

$$\overset{+}{f} + f^- = 0 + 0 = 0 = |f| \qquad \text{(vii)}$$

That is for all f, we have

$$|f| = \overset{+}{f} + f^- \qquad \text{(viii)}$$

Adding (iv) and (viii) we have $f + |f| = 2\overset{+}{f}$,

$$\overset{+}{f} = \tfrac{1}{2}(f + |f|) \qquad \text{(ix)}$$

Similarly on subtracting we obtain $\overset{-}{f} = \tfrac{1}{2}(f - |f|)$ (x)

Since measurability of f implies the measurability of | f | it is obvious from (ix) and (x) that $\overset{+}{f}$ and $\overset{-}{f}$ are measurable.

2.20 Theorem. If f and g are two measurable functions, then f $\vee$g and f $\wedge$g are measurable.

Proof. We know that

$$f\vee g = \frac{f+g+|f-g|}{2}$$

$$f\wedge g = \frac{f+g-|f-g|}{2}$$

Now measurability of f _ measurability of |f|. Also if f and g are measurable, then f+g, f-g are measurable. Hence f$\vee$g and f $\wedge$g are measurable.

2.21 Definition. Characteristic function of a set E is defined by $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$

This is also known as indicator function**.**

2.22 Examples of measurable function

Example. Let E be a set of rationals in [0,1]. Then the characteristic function $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$ **is measurable.**

Proof. For the set of rationals in the given interval, we have $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$

It is sufficient to prove that { x | $\chi_{E(X)} > \alpha$} is measurable for any real α.

Let us suppose first $that\ \alpha \geq 1$. Then { x | $\chi_{E(X)} > \alpha$} ={ x | $\chi_{E(X)} > 1$}

Hence the set { x | $\chi_{E(X)} > \alpha$} is empty in this very case. But outer measure of any empty set is zero. Hence for $\alpha \geq 1$, the set { x | $\chi_{E(X)} > \alpha$} and so $\chi_{E(X)}$ is measurable .

Further let 0$\leq \alpha \leq 1$. Then { x | $\chi_{E(X)} > \alpha$} = E

But E is countable and therefore measurable. Hence $\chi_{E(X)}$ is measurable.

Lastly, let$\alpha \leq 0$. Then { x | $\chi_{E(X)} > \alpha$} = [0,1] and therefore measurable. Hence the result.

2.23 Theorem. Characteristic function χ_A is measurable if and only if A is measurable.

Proof. Let A be measurable. Then $\chi_A(x) = \begin{Bmatrix} 1, x \in A \\ 0, x \notin A \end{Bmatrix}$

Hence it is clear from the definition that domain of χ_A is A $\cup A^c$ which is measurable due to the measurability of A. Therefore, if we prove that the set {x | $\chi_{A(x)} > \alpha$} is measurable for any real α, we are through.

Let $\alpha \geq 0$. Then {x | $\chi_{A(x)} >$ }= {x | $\chi_{A(x)} = 1$}=A(by the definition of Characteristic function.)

But A is given to be measurable. Hence for $\alpha \geq 0$. The set {x | $\chi_{A(x)} > \alpha$ } is measurable.

Now let us take $\alpha < 0$. Then {x | $\chi_{A(x)} >$ }= A∪A^C

Hence {x | $\chi_{A(x)} >$ } is measurable for $\alpha < 0$ also, since A ∪A^C has been proved to be measurable. Hence if A is measurable, then χ_A is also measurable. Conversely, let us suppose that $\chi_{A(x)}$ is measurable. That is,

the set {x | $\chi_{A(x)} > \alpha$ } is measurable for any real α.

Let $\alpha \geq 0$. Then {x | $\chi_{A(x)} >$ }= {x | $\chi_{A(x)} = 1$}= A

Therefore, measurability of{x | $\chi_{A(x)} >$ } implies that of the set A for $\alpha \geq 0$. Now consider $\alpha < 0$. Then {x | $\chi_{A(x)} >$ }= A∪A^C

Thus measurability $\chi_{A(x)}$ of implies measurability of the set A∪A^C which imply A is measurable.

2.24 Simple Function: Let f be a real valued function defined on X. If the range of f is finite. We say that f is a simple function.

Let E⊆X and put $\chi_{E(X)} = \begin{Bmatrix} 1, x \in E \\ 0, x \notin E \end{Bmatrix}$

Suppose the range of f consists of the distinct number $c_1, c_2, \ldots, c_n$.

Let $E_i = \{x: f(x) = c_i\}(i = 1, 2, \ldots, n)$

Then f = $\sum_{i=1}^{n} c_i \chi_{E_i}$

i.e., every simple function is a finite linear combination of characteristic function. It is clear that f is measurable if and only if the sets $E_1, E_2, \ldots, E_n$ are measurable.

2.25 Remarks:

1. **Every step function is a simple function.**
2. **Every simple function is measurable.**
   Proof: Let f be a simple function defined as above.
   Then we have
   f(x) = $\sum_{i=1}^{n} c_i \chi_{E_i}$(x)
   $= c_1\chi_{E_1}(x) + c_2\chi_{E_2}(x) + \cdots + c_n\chi_{E_n}(x)$

$$\therefore f(x) = c_1, x \in E_1$$
$$f(x) = c_2, x \in E_2$$
$$\therefore f(x) = c_i, x \in E_i$$

$$\therefore E_i = \{x: f(x) = c_i\}$$

Since each E_i is measurable. Thus χ_{E_i} is measurable because χ_A is measurable if and only if A is measurable.

Hence f is measurable.

3. **Characteristic function of measurable set is a simple function.**
4. **Product of the simple function and finite linear combination of simple functions is again a simple function.**

2.26 Theorem. (Approximation Theorem). For every non-negative measurable function f, there exists a non-negative non-decreasing sequence $\{f_n\}$ of simple functions such that $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in E$

In the general case if we do not assume non-negativeness of f , then we say For every measurable function f, there exists a sequence $\{f_n\}$, $n \in N$ of simple function which converges (pointwise) to f . i.e. "Every measurable function can be approximated by a sequence of simple functions."

Proof. Let us assume that $f(x) \geq 0$ and $x \in E$. Construct a sequence

$$f_n(x) = \begin{Bmatrix} \frac{i-1}{2^n}, for\ \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} for\ i = 1,2\ \ ,n2^n \\ n, \quad f(x) \geq n \end{Bmatrix} \text{ for every } n \in N.$$

If we take n = 1, then

$$f_1(x) = \begin{Bmatrix} \frac{i-1}{2}, for\ \frac{i-1}{2} \leq f(x) < \frac{i}{2} for\ i = 1,2 \\ 1, \quad f(x) \geq 1 \end{Bmatrix}$$

$$\text{That is, } f_1(x) = \begin{Bmatrix} 0, for\ 0 \leq f(x) < \frac{1}{2} \\ \frac{1}{2}, \quad for\ \frac{1}{2} \leq f(x) < 1 \\ 1\ \ for\ f(x) \geq 1 \end{Bmatrix}$$

Similarly taking n = 2, we obtain

$$f_2(x) = \begin{Bmatrix} \frac{i-1}{4}, for\ \frac{i-1}{4} \leq f(x) < \frac{i}{4}\ for\ i = 1,2,,,8 \\ 2, \quad f(x) \geq 2 \end{Bmatrix}$$

That is,

$$f_2(x) = \begin{Bmatrix} 0\ for\ \ 0 \leq f(x) < \frac{1}{4} \\ \frac{1}{4}\ for\ \frac{1}{4}\ \leq f(x) < \frac{1}{2} \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots .. \\ \frac{7}{4}\ for\ \frac{7}{4}\ \leq f(x) < 2 \\ 2\ for\ f(x) \geq 2 \end{Bmatrix}$$

Similarly we can write $f_3(x)$ (x) and so on. Clearly all f_n are positive whenever f is positive and also it is clear that $f_n \leq f_{n+1}$. Moreover f_n takes only a finite number of values. Therefore $\{f_n\}$ is a sequence of non-negative, non decreasing functions which assume only a finite number of values.

Let us denote

$$E_{ni} = f^{-1}\left[\frac{i-1}{n}, \frac{i}{n}\right] = \left\{x \in E \mid \frac{i-1}{2} \leq f(x) < \frac{i}{2}\right\}$$

and

$$E_n = f^{-1}[n.\infty) = \{x \in E \mid f(x) \geq n\}$$

Both of them are measurable. Let

$$f_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n_i}} + n\chi_{E_n} \quad \text{for every n} \in \text{N} .$$

Now $\sum_{i=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n_i}}$ is measurable, since $\chi_{E_{n_i}}$ has been shown to be measurable and characteristic function of a measurable set is measurable. Similarly χ_{E_n} is also measurable since E_n is measurable. Hence each f_n is measurable. Now we prove the convergence of this sequence.

Let f(x) < ∞. That is f is bounded. Then for some n we have

$$\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}$$

$$\frac{i-1}{2^n} - \frac{i-1}{2^n} \leq f(x) - \frac{i-1}{2^n} < \frac{i}{2^n}$$

$\Rightarrow \quad 0 \leq f(x) - \frac{i-1}{2^n} < \frac{i}{2^n}$

$\Rightarrow \quad 0 \leq f(x) - f_n(x) < \frac{i}{2^n}$ (by the def of f_n (x))

$\Rightarrow \quad f(x) \leq f_n(x) < \varepsilon$

or $|f(x) - f_n(x)| \leq \frac{1}{2^n} < \varepsilon \forall n \geq m \text{ and } x \in E.$

since m does not depend upon point.

Therefore, convergence is uniform.

Let us suppose now that f is not bounded. Then f(x) = ∞

⇨ $f(x) \geq n \text{ for every } n \in N$

But f_n (x) = n

⇨ $\lim_{n\to\infty} f_n(x) = \infty = f(x).$

When **we do not assume non-negativeness of the function** then since we know that $\overset{+}{f}$ and $\overset{-}{f}$ are both non-negative, we have by what we have proved above

$$\overset{+}{f} = \lim_{n\to\infty} \phi_n{}'(x) \qquad \text{(i)}$$

$$\overset{-}{f} = \lim_{n\to\infty} \phi_n{}''(x) \qquad \text{(ii)}$$

where $\emptyset_n{}'(x)$ and $\emptyset_n{}''(x)$ are simple functions. Also we have proved already that

$$\text{f} = \overset{+}{f} - f^-$$

Now from (i) and (ii) we have

$$\begin{aligned} \overset{+}{f} - f^- &= \lim_{n\to\infty} \phi_n{}'(x) - \lim_{n\to\infty} \phi_n{}''(x) \\ &= \lim_{n\to\infty} (\phi_n{}'(x) - \phi_n{}''(x)) \\ &= \lim_{n\to\infty} \phi_n(x) \end{aligned}$$

(since the difference of two simple functions is again a simple function). Hence the theorem.

We now introduce the terminology "almost everywhere" which will be frequently used in the Sequel.

2.27 Definition. A statement is said to hold almost everywhere in E if and only if it holds everywhere in E except possibly at a subset D of measure zero.

(a) Two functions f and g defined on E are said to be equal almost everywhere in E iff f(x) =g(x) everywhere except a subset D of E of measure zero.
(b) A function defined on E is said to be continuous almost everywhere in E if and only if there exists a subset D of E of measure zero such that f is continuous at every point of E□D.

2.28 Theorem. (a) If f is a measurable function on the set E and $E_1 \subseteq E$is measured set, then f is a measurable function on E_1.

(b) If f is a measurable function on each of the sets in a countable collection{E_i } of disjoint measurable sets, then f is measurable.

Proof. (a) For any real α, we have {x ∈ E_1, f(x) > α } = { x ∈E; f(x) > α } ∩E1. The result follows as the set on the right-hand side is measurable.

(b)Write E = $\bigcup_{i=1}^{\infty} E_i$, Clearly, E, being the union of measurable set is measurable. The result now follows, since for each real α, we have

E = { x ∈E: f(x) > α } = { x ∈$\bigcup_{i=1}^{\infty} E_i$: f(x) > α }

2.29 Theorem. Let f and g be any two functions which are equal almost everywhere in E. If f is measurable so is g.

Proof. Since f is measurable, for any real, the set {x | f(x) > } is measurable. We shall show that the set {x | g(x) > } is measurable. To do so we put

E_1 = {x | f(x) > } and E_2 = {x | g(x) > α }. Consider the sets

$E_1 - E_2$ and $E_2 - E_1$.

These are subsets of $\{x: f(x) \neq g(x)\} \begin{bmatrix} \because x \in E_1 - E_2 \Longrightarrow x \in E_1 and\ x \notin E_2 \\ f(x) > \alpha, g(x) \not> \alpha \Longrightarrow f(x) \neq g(x) \end{bmatrix}$

But f = g a.e.

⇨ m$\{x: f(x) \neq g(x)\}$ = 0

$$E_1 - E_2 \subseteq \{x: f(x) \neq g(x)\}$$

m$(E_1 - E_2) \leq$ m$\{x: f(x) \neq g(x)\}$ = 0

⇨ m$(E_1 - E_2) \leq 0$ But m$(E_1 - E_2) \geq 0$

⇨ m$(E_1 - E_2) = 0$

Similarly m$(E_2 - E_1)$ =0

∴ m$(E_2 - E_1)$ = 0 = m$(E_1 - E_2)$

⇨ $(E_1 - E_2)$ and $(E_2 - E_1)$ are measurable.

⇨ $E_2 = [E_1 \cup (E_2 - E_1)] - (E_1 - E_2)$

Since E_1, $E_2 - E_1$ and $(E_1 - E_2)^C$ are measurable therefore it follows that E2 is measurable. Hence the theorem is proved.

2.30. Corollary. Let $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n = f$ almost everywhere. Then f is a measurable function.

Proof. We have already proved that if $\{f_n\}$ is a sequence of measurable functions then $\lim_{n\to\infty} f_n$ is measurable. Also, it is given that $\lim_{n\to\infty} f_n$ = f a.e. Therefore, using the above theorem, it follows that f is measurable.

2.31 Definition: (Restriction of f to E_1)

Let f be a function defined on E, then the function f_1 defined on E_1 contained in E .i.e., $E_1 \subseteq E$ by $f_1(x) = f(x)$, $x \in E_1$ is called restriction of f to E_1 and denoted by f/ E_1.

2.32 Exercise : Let f be a measurable function defined on E, then its restriction to E_1 is also measurable where E_1 is a measurable subset of E.

Solution : Let f_1 = f/ E_1 i.e., f_1 is restriction of f to E_1.

Let α *be any real number*.

$\{x \in E_1: f_1(x) > \alpha\} = \{x \in E_1: f(x) > \alpha\}$ $[\because f_1 = f \ on \ E_1]$

= { x ∈E: f(x) > α } ∩ E_1 is measurable on E and E_1 is also measurable and intersection of measurable sets is measurable. Hence f_1 is measurable on E_1.

2.33Exercise:Let f be a measurable function defined on $\boldsymbol{where\ E_1\ and\ E_2\ are\ measurable.}$ $\boldsymbol{Then\ the\ function\ f\ is\ measurable\ on\ E_1 U E_2\ iff\ \frac{f}{E_1} and \frac{f}{E_2}\ are\ measurable.}$

Solution: Let f_1 = f/ E_1 and Let f_2 = f/ E_2

Let E = $E_1 U E_2$

Clearly E is measurable because $E_1\ and\ E_2\ are\ measurable$. Suppose f is measurable on E then by previous exercise f_1 is measurable on E_1 and f_2 is measurable on E_2.

Conversely, Let $\alpha\ be\ any\ real\ number$.

Therefore

{x ∈E: f(x) > α } = { x ∈ $E_1 U E_2$: f(x) > α }

= { x ∈E_1: f(x) > α }∪ { x ∈E_2: f(x) > α }

= { x ∈E_1: f_1(x) > α }∪ { x ∈E_2: f_2(x) > α }

because f_1 is measurable on E_1 and f_2 is measurable on E_2.

- ⇨ f is measurable on E = $E_1 U E_2$.

2.24 Theorem. If a function f is continuous almost everywhere in E, then f is measurable.

Proof. Since f is continuous almost everywhere in E, there exists a subset D of E with m*D = 0 such that f is continuous at every point of the set C = E-D.

To prove that f is measurable, let α denote any given real number.

Consider the set {x ∈E | f(x) > } = B(say)

We have to show that B is measurable. If B ∩C = $\varphi, then\ B \subseteq D$.

- ⇨ m*(B) ≤ m*(D) = 0.
- ⇨ m*(B) = 0.
- ⇨ B is measurable.

Now suppose that B ∩C ≠ φ. For this purpose, let x denote an arbitrary point in B ∩C. Then x ∈ $B\ and\ x \in C$ ⟹f(x) > α and f is continuous at x. Hence there exists an open interval U_x containing x such that f(y) > α hold for every point y of E∩ Ux. Let U =$\bigcup_{x \in B \cap C} U_x$.

Since x ∈E ∩Ux ⊂B holds for every x ∈B ∩C, we have

B ∩C⊂E ∩Ux ⊂B. This implies B = (E ∩U) ∪(B ∩D). As an open subset of R, U is measurable. Hence E ∩U is measurable. On the other hand, since m*(B∩D) ≤m*D = 0 ,B∩D is also measurable. This implies that B is measurable. This completes the proof of the theorem.

2.25 Littlewood's three principles of measurability

The following three principles concerning measure are due to Littlewood.

First Principle. Every measurable set is a finite union of intervals.

Second Principle. Every measurable function is almost a continuous function.

Third Principle. If $\{f_n\}$ is a sequence of measurable function defined on a set E of finite measure and if $f_n(x) \to f(x)$ on E, then $f_n(x)$ converges almost uniformly on E.

First of all we consider third principle. We shall prove Egoroff's theorem which is a slight modification of third principle of Littlewood's.

2.26 Theorem. Let E be a measurable set with finite measure and $\{f_n\}$ be a sequence of measurable functions defined on a set E such that

$$f_n(x) \to f(x) \text{ for each } x \in E.$$

Then given $\varepsilon > 0$ and $\delta > 0$, there corresponds a measurable subset A of E with m(A) < δ and an integer N such that $|f_n(x) - f(x)| < \varepsilon \;\forall\, x \in E - A \text{ and } n \geq N.$

Proof: Consider the sets $G_n = \{x \in E: |f_n(x) - f(x)| \geq \varepsilon\}$

Now since f_n and f are measurable.

So the sets $G_n{}'s$ are also measurable.

Now define $E_k = \bigcup_{n=k}^{\infty} G_n$.

$$= \{x: x \in G_n \text{ for some } n \geq k\}$$

$$= \{x: x \in E, |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq k\}$$

We observe that $E_{k+1} \subseteq E_k$.

On the contrary, we assume that for each $x \in E_k \forall\, k$.

Then for any fixed given k, we must have

$$E_k = \{|f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq k\}$$

But this leads to $f_n(x) \nrightarrow f(x)$. a contradiction.

Hence for each $x \in E$ there is some E_k such that $x \notin E_k \Longrightarrow \bigcap_{k=1}^{\infty} E_k = \emptyset$

Now measure of E is finite, so by proposition of decreasing sequence, we have

$$\lim_{n\to\infty} m(E_n) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\emptyset) = 0$$

$$\lim_{n\to\infty} m(E_n) = 0.$$

Hence given $\delta > 0$, $\exists$ an integer N such that $m(E_k) < \delta \forall\, k \geq N$.

In particular put k = N

$$m(E_n) < \delta$$

$$m\{x: x \in E, |f_n(x) - f(x)| \geq \varepsilon \text{for some } n \geq N\} < \delta$$

If we write A = E_n, then m(A) $< \delta$ *and*

E-A = $\{x: x \in E, |f_n(x) - f(x)| < \varepsilon \text{for all } n \geq N\}$

In other words,

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N \text{ and } x \in E - A$$

This completes the proof.

2.27 Definition: A Sequence {fn} of functions defined on a set E is said to converge almost everywhere to f if $\lim_{n\to\infty} f_n(x) = f(x)\ \forall\, x \in E - E_1$ *where* $E_1 \subset E$,

$$m(E_1) = 0.$$

2.28 Theorem. Let E be a measurable set with finite measure and {fn} be a sequence of measurable functions converging almost everywhere to a real valued function f defined on a set E. Then given $\varepsilon > 0$ and $\delta > 0$, there corresponds a measurable subset A of E with m(A) < δ and an integer N such that $|f_n(x) - f(x)| < \varepsilon\ \forall\, x \in E - A$ and $n \geq N$.

Proof: Let F be a set of points of E for which $f_n(x) \nrightarrow$f. Then m(F) = 0.

Since $f_n(x) \to f(x)$ almost everywhere, then

$f_n(x) \to f(x)\ \forall\, x \in E - F = E_1 (say)$

Now applying the last theorem for the set E_1, we get a set $A_1 \subseteq E_1$ with $m(A_1) < \delta$ and an integer N such that $|f_n(x) - f(x)| < \varepsilon\ \forall\, x \in E_1 - A_1$ and $n \geq N$.

Now the required result follows if we take

$$A = A_1 \cup F \text{ as shown below.}$$

$$m(A) = m(A_1 \cup F) = m(A_1) + m(F) = m(A_1) + 0 = m(A_1) < \delta$$

Also $E - A = E - (A_1 \cup F) = E \cap (A_1 \cup F)^c$

$= E \cap {A_1}^c \cap F^c = (E \cap F^c) \cap {A_1}^c$

$= (E - F) \cap {A_1}^c = E_1 \cap {A_1}^c = E_1 - A_1$

i.e., E-A = $E_1 - A_1$

Hence we have found a set A ⊆ E with m(A) < δ and an integer N such that $|f_n(x) - f(x)| < \varepsilon\ \forall\, x \in E - A$ and $n \geq N$.

2.29 Definition: A Sequence {fn} of functions is said to converge almost uniformly everywhere to a measurable function f defined on a measurable set E if for each ε >0, ∃ *a measurable set* A ⊆ E with m(A) < ε such that and an integer N such that f_n *converges to f uniformly on* $E - A$.

2.30 Theorem.(Egoroff's Theorem). Let {fn} be a sequence of measurable functions defined on a set E of finite measure such that $f_n(x) \to f(x)$ almost everywhere. Then to each η> 0 there corresponds a measurable subset A of E such that m (A) < η such that fn(x) converges to f(x) uniformly on E-A.

Proof. Applying last theorem with $\varepsilon = 1, \delta = \eta/2$

We get a measurable subset $A_1 \subseteq E$ with $m(A_1) < \eta/2$ and positive integer N_1 such that

$$|f_n(x) - f(x)| < 1 \text{ for all } n \geq N_1 \text{ and } x \in E_1 (= E - A_1)$$

Again taking $\varepsilon = \frac{1}{2}, \delta = \eta/2^2$

We get another measurable subset $A_2 \subseteq E_1$ with $m(A_2) < \eta/2^2$ and positive integer N_2 such that

$$|f_n(x) - f(x)| < \frac{1}{2} \; for \; all \; n \geq N_2 \; and \; x \in E_2 (= E_1 - A_2)$$

Continuing like that at kth stage, we get a measurable subset $A_k \subseteq E_{k-1}$ with

$m(A_k) < \eta/2^k$ and positive integer N_k such that

$$|f_n(x) - f(x)| < \frac{1}{k} \; for \; all \; n \geq N_k \; and \; x \in E_k (= E_{k-1} - A_k)$$

Now we set A = $\bigcup_{k=1}^{\infty} A_k$

Then we have

m(A) $\leq \sum_{k=1}^{\infty} m(A_k) < \sum_{k=1}^{\infty} \eta/2^k = \eta . \sum_{k=1}^{\infty} \frac{1}{2^k} = \eta.$

Also E-A = E- $\bigcup_k A_k = \bigcap_k [E_{k-1} - A_k] = \bigcap_k E_k [\because E_{k-1} - A_k = E_k]$

Let $x \in E - A, then \; x \in E_k \forall \; k \; and \; so \; |f_n(x) - f(x)| < \frac{1}{k} \forall n \geq N_k.$

Choose k such that $\frac{1}{k} < \varepsilon \; so \; that \; we \; get$

$|f_n(x) - f(x)| < \varepsilon \; \forall \; x \in E - A \; and \; n \geq N_k = N.$

This completes the proof of the theorem.

Now we pass to the **second principle of Littlewood**. This is nothing but approximation of measurable functions by continuous functions. In this connection we shall prove the following theorem known as Lusin Theorem after the name of a Russian Mathematician Lusin, N.N.

2.31 Lusin Theorem: Let f be a measurable real valued function defined on closed interval [a,b], then given $\boldsymbol{\delta} > 0, \exists \; a \; continuous \; function \; g \; on \; [\boldsymbol{a}, \boldsymbol{b}] \boldsymbol{such \; that}$

$$\boldsymbol{m}\{\boldsymbol{x} : \boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{g}(\boldsymbol{x})\} < \delta.$$

Proof: First we prove two lemmas.

Lemma 1. Let F be a closed subset of R, then a function g: F→ $\boldsymbol{R}$ is continuous if sets $\{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{a}\} \; \boldsymbol{and} \; \{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) \geq \boldsymbol{b}\}$ are closed subsets of F for every rational a and b.

Proof: Let $\{x : g(x) \leq a\} \; and \; \{x : g(x) \geq b\} \; are \; closed \; subset \; of \; F.$

$\Rightarrow \{x : g(x) > a\} \cap \{x : g(x) < b\}$ is open subset of F.

.i.e., $\{x : a < g(x) < b\}$ is open.

i.e., $\{x : g(x) \in (a, b)\} \; is \; open \; in \; F.$

i.e., $g^{-1}(a, b) is \; open \; in \; F.$

Let O be any open set in R then O can be written as countable union of disjoint open intervals with rational end points.

Let O = $\bigcup_{n=1}^{\infty}(a_n, b_n)$

Then g^{-1}O = $g^{-1}(\bigcup_{n=1}^{\infty}(a_n, b_n))$ = $\bigcup_{n=1}^{\infty} g^{-1}(a_n, b_n)$

Since $g^{-1}(a,b) is\ open$ and countable union of open set is open.

$\Rightarrow g^{-1}(O)\ is\ open \Rightarrow g\ is\ continuous.$

Lemma 2. Let f: [a, b] $\rightarrow \boldsymbol{R}$ be a measurable function, then given $\boldsymbol{\delta} > 0, \exists\ a\ closed\ subset\ F\ of\ E = [\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{such\ that\ m}(\boldsymbol{E} - \boldsymbol{F}) < \ \delta\ and \frac{\boldsymbol{f}}{\boldsymbol{F}}\ \boldsymbol{is\ continuous}.$

Proof: Let $\{r_n\}\ be\ a\ sequence\ of\ all\ rational\ numbers.$

For n$\in N$, take $A_n = \ \{x: f(x) \geq r_n\}$

And $A_n{}^* = \{x: f(x) \leq r_n\}$

Clearly each $A_n\ and A_n{}^*\ are\ measurable\ [\because\ f\ is\ measurable]$

Then $\exists$ closed sets $B_n \subset A_n\ and\ B_n{}^* \subset A_n{}^*\ such\ that$

$$m(A_n - B_n) < \frac{\delta}{2^n.3}\ and\ m(A_n{}^* - B_n{}^*) < \frac{\delta}{2^n.3}$$

Let $D = [\bigcup_{n=1}^{\infty}(A_n - B_n)] \cup [\bigcup_{n=1}^{\infty}(A_n{}^* - B_n{}^*)]$

Clearly D is measurable.

Therefore m(D) $\leq \sum_{n=1}^{\infty} m(A_n - B_n) + \sum_{n=1}^{\infty} m(A_n{}^* - B_n{}^*)$

m(D) < $\sum_{n=1}^{\infty} \frac{\delta}{2^n.3} + \sum_{n=1}^{\infty} \frac{\delta}{2^n.3}$

$= \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3}$

$$\Rightarrow m(D) < \frac{2\delta}{3}.$$

Now E and D are measurable.

- ⇨ E-D is measurable.

  Then for given $\delta > 0, \exists\ a\ closed\ set\ F \subseteq E - D\ such\ that\ m(E - D - F) < \frac{\delta}{3}$

  Now E-F = D$\cup$ $(E - F - D)$

- ⇨ m(E-F) = m(D) + m$(E - F - D) < \frac{2\delta}{3} + \frac{\delta}{3} = \ \delta$

  Let h = f/F

  To show that h is continuous on F.

  For rational number r_n,

$$\{x: h(x) \leq r_n\} = \{x: f(x) \leq r_n\} \cap F$$
$$= A_n{}^* \cap F = \left[\left((A_n{}^* - B_n{}^*) \cup B_n{}^*\right)\right] \cap F$$
$$=\left[\left((A_n{}^* - B_n{}^*) \cap F\right)\right] \cup [B_n{}^* \cap F]$$
$$= \emptyset \cup [B_n{}^* \cap F]$$
$$= B_n{}^* \cap F$$

$$D = \left[\bigcup_{n=1}^{\infty}(A_n - B_n)\right] \cup \left[\bigcup_{n=1}^{\infty}(A_n{}^* - B_n{}^*)\right]$$

$$\Rightarrow (A_n{}^* - B_n{}^*) \subset D$$

$$\because\ F \subseteq E - D \Rightarrow F \cap D = \emptyset.$$

$$\{x: h(x) \leq r_n\} = B_n{}^* \cap F$$

Since $B_n{}^*$ *is closed in* $E = [a, b]$.

$$B_n{}^* \cap F \text{ is closed in } F.$$

⇨ $\{x: h(x) \leq r_n\}$ *is closed in* F.

By lemma 1, h is continuous.

So f/F is continuous.

Lusin Theorem:(Proof):- We have

f:[a, b]→ R is measurable function, then by lemma(2), for given $\delta > 0, \exists$ *a closed set* $F \subset E$ *such that* $m(E - F) < \delta$ *and* $h = \frac{f}{F}$ *is continuous*.

Now using result "Every real valued continuous function defined on a closed subset of a real number can be extended continuously to all real numbers."

So h can be extended to continuous function h*: R→ R.

Let g :[a, b]→ R, g *is continuous*

$$\text{and for } x \in F, g(x) = f(x) \text{ on } F.$$

$$\text{and } \{x \in E: f(x) \neq g(x)\} \subseteq E - F$$

$$m\{x \in E: f(x) \neq g(x)\} \leq m(E - F) < \delta.$$

"Convergence in Measure"

The notion of convergence in measure is introduced by F.Reisz and E.Fisher in 1906-07. Sometimes it is also called approximate convergence.

2.32 Definition. A sequence < f_n > of measurable functions is said to convergence in measure to f on a set E, written as $f_n \xrightarrow{m} f$ *on* E,

If given δ> 0, $\exists\ m \in$ N such that for all n $\geq m$, we have

$$m\{x||f(x) - f_n(x)| \geq \varepsilon\} < \delta.$$

Or $\lim_{n\to\infty} m\{x||f(x) - f_n(x)| \geq \varepsilon\} = 0$

This means that for all sufficiently large value of n, functions f_n of the sequence < fn > differ from the limit function f by a small quantity with the exception of the set of point whose measure is arbitrary small (<δ).

2.33 Theorem: If sequence $\{f_n\}$ converges in measure to the function *f*, then it converges in measure to every function *g* which is equivalent to the function.

Proof: For each $\varepsilon > 0$, we have

$$\{x: |f_n(x) - g(x)| \geq \varepsilon\} \subset \{x: f(x) \neq g(x)\} \cup \{x: |f_n(x) - f(x)| \geq \varepsilon\}$$

Since g is equivalent to f, then we have

$$m\{x: f(x) \neq g(x)\} = 0.$$

$$m\{x: |f_n(x) - g(x)| \geq \varepsilon\} \leq m\{x: f(x) \neq g(x)\} + m\{x: |f_n(x) - f(x)| \geq \varepsilon\}$$
$$\leq m\{x: |f_n(x) - f(x) \geq \varepsilon|\} < \delta$$

$$\Rightarrow \boldsymbol{f_n \xrightarrow{m} g}.$$

Hence the result.

2.34 Theorem: If sequence $\{f_n\}$ converges in measure to the function *f*, then the limit function *f* is unique a.e.

Proof: Let g be another function such that $\boldsymbol{f_n \xrightarrow{m} g}$.

Since $|f - g| \leq |f - f_n| + |f_n - g|$

Now we observe that for each $\varepsilon > 0$,

$$\{x: |f(x) - g(x)| \geq \varepsilon\} \subset \left\{x: |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x: |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}$$

Since by proper choice of ε, the measure of both the sets on the right can be made arbitrary small, we have

$m\{x: |f(x) - g(x)| \geq \varepsilon\}$=0

- ⇨ f = g almost everywhere. Hence the proof.

2.35 Theorem: Let $\{f_n\}$ be a sequence of measurable functions which converges to f a.e. on X. Then $f_n \xrightarrow{m} f$ *on X*.

Proof: For each n∈ N *and* $\varepsilon > 0$, *Consider the sets*

$$S_n(\varepsilon) = \{x \in X: |f_n(x) - f(x)| \geq \varepsilon\}$$

Let $\delta > 0$ *be any arbitrary number, then* ∃*a measurable set* $A \subset X$

With m(A) < δ and the number N such that

$$|f_n(x) - f(x)| < \varepsilon \; \forall x \in X - A \text{ and } n \geq N$$

Then it follows that $S_n(\varepsilon) \subset A \; \forall \; n \geq N$

- ⇨ $m(S_n(\varepsilon)) < m(A) < \delta \forall \; n \geq N$
- ⇨ $\lim_{n\to\infty} m(S_n(\varepsilon)) = 0$

Hence$\boldsymbol{f_n \xrightarrow{m} f}$ ***on X***.

2.36 Remark: The converse of the above theorem need not be true i.e, convergence in measure is more general than a.e. infact there are sequence of measurable functions that converges in measure but fails to converge at any point.

To affect we consider the following example

$$\boldsymbol{f_n: [0, 1] \to R \; as}$$

$$\boldsymbol{f_n(x) = \begin{cases} 1, if \; x \in \left[\frac{k}{2^t}, \frac{k+1}{2^t}\right] \\ 0, otherwise \end{cases}}$$

Let n = $k + 2^t$ *where* $0 \leq k \leq 2^t$.

Let $\varepsilon > 0$ *be given*. Choose an m $\in N$ *such that* $\frac{2}{m} < \varepsilon$

Then $m\{x: |f_n(x) - 0| \geq \varepsilon\} = m\{x: |f_n| \geq \varepsilon\}$

$$= \frac{1}{2^t} < \frac{1}{2^n} \left[\begin{matrix} \because n = k + 2^t < 2^t + 2^t \\ < 2.2^t, \ \frac{1}{2^t} < \frac{2}{n} \end{matrix} \right] \qquad (*)$$

$$\leq \frac{2}{m} < \varepsilon \ \forall n \geq m$$

- ⇨ f_n converges in measure to zero for x∈ $[0, 1]$
- ⇨ $i.e., f_n \overset{m}{\rightarrow} [0, 1]$
  f_n (x) has value 1 for arbitrary large value of n and so it does not converge to zero a.e. because on taking n very large, we get 2^t large and hence number of subintervals of type (*) increase and possibility of $f_n(x) = 1$ is more.

2.37 Theorem (F. Riesz). "Let < fn > be a sequence of measurable functions which converges in measure to f. Then there is a subsequence < f_{n_k} > of < fn > which converges to **f almost everywhere**."

Proof. Let $\boldsymbol{f_n \overset{m}{\rightarrow} f}$.

Let us consider two sequences $\left\{\frac{1}{n}\right\}$ *and* $\left\{\frac{1}{2^n}\right\}$ *of real numbers such that*

$$\frac{1}{n} \rightarrow 0 \ as\ n \rightarrow \infty \ as \ \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

We now choose a strictly increasing sequence $\{n_k\}$ of positive integer as follows

Let n_1 be a positive integer such that

$$m\left(\left\{x: \left|f_{n_1}(x) - f(x)\right| \geq 1\right\}\right) < \frac{1}{2}$$

Such a number n_1 exists since in view $\boldsymbol{f_n \overset{m}{\rightarrow} f}$ for a given $\varepsilon_1 = 1 > 0$ *and*

$\delta_1 = \frac{1}{2} > 0, \exists$ *an integer* n_1*such that*

$$m(\{x: |f_n(x) - f(x)| \geq 1\}) < \frac{1}{2} \forall n \geq n_1$$

In particular for n = n_1.

Similarly, Let n_2 be a positive number such that $n_2 \geq n_1$ *and*

$$m\left(\left\{x: \left|f_{n_2}(x) - f(x)\right| \geq \frac{1}{2}\right\}\right) < \frac{1}{2^2} \ and\ so\ on.$$

Continuing in this process, we get the positive number$n_k \geq n_{k-1}$

$$m\left(\left\{x: \left|f_{n_k}(x) - f(x)\right| \geq \frac{1}{k}\right\}\right) < \frac{1}{2^k}$$

Now set $E_k = \bigcup_{i=k}^{\infty}\left\{x: \left|f_{n_i}(x) - f(x)\right| \geq \frac{1}{i}\right\}, k \in N.$

And $E = \bigcap_{k=1}^{\infty} E_k$

Then it is clear that $\{E_k\}$ is decreasing sequence of measurable sets.

Therefore $m(E) = \lim_{k\to\infty} m(E_k)$

But $m(E_k) = m\left\{\bigcup_{i=k}^{\infty}\left\{x: \left|f_{n_i}(x) - f(x)\right| \geq \frac{1}{i}\right\}\right\}$

$\leq \sum_{i=k}^{\infty} m\left\{x: \left|f_{n_i}(x) - f(x)\right| \geq \frac{1}{i}\right\}$

$< \sum_{i=k}^{\infty} \frac{1}{2^i} \to 0 \; as \; k \to \infty$

$= \frac{1}{2^{k-1}}$

Hence m(E) = 0.

Thus it remains to be verified that the sequence $< f_{n_k} >$ converges to f on X-E.

So let $x_0 \notin E$. Then $x_0 \notin E_m$ for some positive integer m.

i.e., $x_0 \notin \left\{x: \left|f_{n_k}(x) - f(x)\right| \geq \frac{1}{k}\right\}, k \geq m$

$$\Rightarrow \left|f_{n_k}(x) - f(x)\right| < \frac{1}{k}, k \geq m$$

But $\frac{1}{k} \to 0$ as k$\to \infty$

Hence $\lim_{k\to\infty} f_{n_k}(x_0) = f(x_0)$.

Since $x_0 \in X - E$ was arbitrary, it follows that

$\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for each x $\in X - E$ and so $\left\{f_{n_k}\right\}$ $converges \; to \; f \; a.e.$

This completes the proof.